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A Study on Matrix Fractional Exponential Functions

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Abstract: **In this paper, we find the closed forms of four matrix fractional exponential functions. In fact, our results are generalizations of ordinary calculus results.**

Keywords: **closed forms, matrix fractional exponential functions.**

I. INTRODUCTION

Fractional calculus is a natural extension of the traditional calculus. In fact, since the beginning of the theory of differential and integral calculus, some mathematicians have studied their ideas on the calculation of non-integer order derivatives and integrals. During the 18th and 19th centuries, there were many famous scientists such as Euler, Laplace, Fourier, Abel, Liouville, Grunwald, Letnikov, Riemann, Laurent, Heaviside, and some others who reported interesting results within fractional calculus. With the development of computer technology, fractional calculus is widely used in various fields of science and engineering, such as physics, mechanics, electrical engineering, viscoelasticity, economics, bioengineering, and control theory [1-15].

In this paper, we find the closed forms of the following four matrix fractional exponential functions:

$$E_\alpha\left(\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}x^\alpha\right),$$

$$E_\alpha\left(\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}x^\alpha\right),$$

$$E_\alpha\left(\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}x^\alpha\right),$$

$$E_\alpha\left(\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}x^\alpha\right),$$

where $0 < \alpha \leq 1$. In fact, our results are generalizations of classical calculus results.

II. PRELIMINARIES

Definition 2.1 ([16])**:** If x, x_0, and a_n are real numbers for all n, $x_0 \in (a, b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a, b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a, b] \to R$ is continuous on closed interval $[a, b]$ and it is α-fractional analytic at every point in open interval (a, b), then f_α is called an α-fractional analytic function on $[a, b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

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Definition 2.2 ([17])**:** If $0 < \alpha \leq 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha}, \tag{1}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha}. \tag{2}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)} \left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m \right) (x - x_0)^{n\alpha}. \end{aligned} \tag{3}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m \right) \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^{\otimes_\alpha n}. \end{aligned} \tag{4}$$

Definition 2.3 ([18])**:** If $0 < \alpha \leq 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha n}. \tag{5}$$

On the other hand, the α-fractional cosine and sine function are defined as follows:

$$\cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha 2n}, \tag{6}$$

and

$$\sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha (2n+1)}. \tag{7}$$

Moreover, the α-fractional hyperbolic cosine and hyperbolic sine function are defined as follows:

$$\cosh_\alpha(x^\alpha) = \frac{1}{2}[E_\alpha(x^\alpha) + E_\alpha(-x^\alpha)] = \sum_{n=0}^{\infty} \frac{x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha 2n}, \tag{8}$$

and

$$\sinh_\alpha(x^\alpha) = \frac{1}{2}[E_\alpha(x^\alpha) - E_\alpha(-x^\alpha)] = \sum_{n=0}^{\infty} \frac{x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha (2n+1)}, \tag{9}$$

Definition 2.4: If $0 < \alpha \leq 1$, and A is a matrix. The matrix α-fractional exponential function is defined by

$$E_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(A \frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha n}. \tag{10}$$

And the matrix α-fractional cosine and sine function are defined as follows:

$$\cos_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^n \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(A \frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha 2n}, \tag{11}$$

and

$$\sin_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^n \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(A \frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha (2n+1)}. \tag{12}$$

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In addition, the matrix α-fractional hyperbolic cosine and hyperbolic sine function are defined as follows:

$$cosh_\alpha(Ax^\alpha) = \frac{1}{2}[E_\alpha(Ax^\alpha) + E_\alpha(-Ax^\alpha)] = \sum_{n=0}^{\infty} A^{2n}\frac{x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \quad (13)$$

and

$$sinh_\alpha(Ax^\alpha) = \frac{1}{2}[E_\alpha(Ax^\alpha) - E_\alpha(-Ax^\alpha)] = \sum_{n=0}^{\infty} A^{2n+1}\frac{x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}, \quad (14)$$

III. MAIN RESULTS

In this section, we find the closed forms of four matrix fractional exponential functions.

Theorem 3.1: *If* $0 < \alpha \le 1$, *then*

$$E_\alpha\left(\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}x^\alpha\right) = \begin{bmatrix}cos_\alpha(x^\alpha) & -sin_\alpha(x^\alpha)\\ sin_\alpha(x^\alpha) & cos_\alpha(x^\alpha)\end{bmatrix}, \quad (15)$$

$$E_\alpha\left(\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}x^\alpha\right) = \begin{bmatrix}cos_\alpha(x^\alpha) & sin_\alpha(x^\alpha)\\ -sin_\alpha(x^\alpha) & cos_\alpha(x^\alpha)\end{bmatrix}, \quad (16)$$

$$E_\alpha\left(\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}x^\alpha\right) = \begin{bmatrix}cosh_\alpha(x^\alpha) & sinh_\alpha(x^\alpha)\\ sinh_\alpha(x^\alpha) & cosh_\alpha(x^\alpha)\end{bmatrix}, \quad (17)$$

$$E_\alpha\left(\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}x^\alpha\right) = \begin{bmatrix}cosh_\alpha(x^\alpha) & -sinh_\alpha(x^\alpha)\\ -sinh_\alpha(x^\alpha) & cosh_\alpha(x^\alpha)\end{bmatrix}. \quad (18)$$

Proof $E_\alpha\left(\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}x^\alpha\right)$

$$= \sum_{n=0}^{\infty}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)}$$

$$= I + \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\frac{x^\alpha}{\Gamma(\alpha+1)} + \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}^2\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}^3\frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}^4\frac{x^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots$$

$$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\frac{x^\alpha}{\Gamma(\alpha+1)} + \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\frac{x^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots$$

$$= \begin{bmatrix}1 - \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{4\alpha}}{\Gamma(4\alpha+1)} - \cdots & -\frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} - \cdots\\ \frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots & 1 - \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{4\alpha}}{\Gamma(4\alpha+1)} - \cdots\end{bmatrix}$$

$$= \begin{bmatrix}cos_\alpha(x^\alpha) & -sin_\alpha(x^\alpha)\\ sin_\alpha(x^\alpha) & cos_\alpha(x^\alpha)\end{bmatrix}.$$

And

$$E_\alpha\left(\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}x^\alpha\right)$$

$$= \sum_{n=0}^{\infty}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)}$$

$$= I + \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\frac{x^\alpha}{\Gamma(\alpha+1)} + \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}^2\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}^3\frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}^4\frac{x^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots$$

$$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\frac{x^\alpha}{\Gamma(\alpha+1)} + \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\frac{x^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots$$

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$$=\begin{bmatrix} 1-\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}-\cdots & \frac{x^{\alpha}}{\Gamma(\alpha+1)}-\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots \\ -\frac{x^{\alpha}}{\Gamma(\alpha+1)}+\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}-\cdots & 1-\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}-\cdots \end{bmatrix}$$

$$=\begin{bmatrix} cos_\alpha(x^\alpha) & sin_\alpha(x^\alpha) \\ -sin_\alpha(x^\alpha) & cos_\alpha(x^\alpha) \end{bmatrix}.$$

And

$$E_\alpha\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x^\alpha\right)$$

$$=\sum_{n=0}^{\infty}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)}$$

$$=I+\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{x^{\alpha}}{\Gamma(\alpha+1)}+\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^3\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}+\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^4\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\cdots$$

$$=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{x^{\alpha}}{\Gamma(\alpha+1)}+\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}+\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\cdots$$

$$=\begin{bmatrix} 1+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\cdots & \frac{x^{\alpha}}{\Gamma(\alpha+1)}+\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots \\ \frac{x^{\alpha}}{\Gamma(\alpha+1)}+\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots & 1+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\cdots \end{bmatrix}$$

$$=\begin{bmatrix} cosh_\alpha(x^\alpha) & sinh_\alpha(x^\alpha) \\ sinh_\alpha(x^\alpha) & cosh_\alpha(x^\alpha) \end{bmatrix}.$$

Finally,

$$E_\alpha\left(\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} x^\alpha\right)$$

$$=\sum_{n=0}^{\infty}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)}$$

$$=I+\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\frac{x^{\alpha}}{\Gamma(\alpha+1)}+\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}^2\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}^3\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}+\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}^4\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\cdots$$

$$=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\frac{x^{\alpha}}{\Gamma(\alpha+1)}+\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}+\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\cdots$$

$$=\begin{bmatrix} 1+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\cdots & -\frac{x^{\alpha}}{\Gamma(\alpha+1)}-\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}-\cdots \\ -\frac{x^{\alpha}}{\Gamma(\alpha+1)}-\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}-\cdots & 1+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\cdots \end{bmatrix}$$

$$=\begin{bmatrix} cosh_\alpha(x^\alpha) & -sinh_\alpha(x^\alpha) \\ -sinh_\alpha(x^\alpha) & cosh_\alpha(x^\alpha) \end{bmatrix}.$$

q.e.d.

IV. CONCLUSION

In this paper, we find the closed forms of four matrix fractional exponential functions. In fact, our results are generalizations of classical calculus results. In the future, we will continue to use matrix fractional exponential functions to solve problems in engineering mathematics and fractional differential equations.

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